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# An energy approach to the proof of the existence of Rayleigh waves in an anisotropic elastic half-space☆

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ABSTRACT

An approach based on investigating the energy functional is applied for the first time to the classical problem of Rayleigh waves in an anisotropic half-space with a free boundary. The main object of the investigation is an ordinary differential operator in a variable characterizing the depth. An investigation of the spectrum by variational methods enables a new proof to be given of the existence of a Rayleigh wave in a linear elastic half-space with arbitrary anisotropy, which does not rest on the Stroh formalism.



## 1. Introduction

Rayleigh[1] discovered that a non-dispersive surface wave – a linear combination of two non-uniform plane waves, longitudinal and transverse, can propagate on the free surface of a homogeneous isotropic elastic half-space, travelling along the boundaries more slowly than the corresponding uniform waves and hence attenuating exponentially with depth. Rayleigh's arguments (which were given a rigorous form in Ref. 2) where based on an explicit form of the dispersion equation.

In the Twentieth Century a large number of papers were devoted to surface waves. The extension of the Rayleigh wave to the an isotropic case (known as a Rayleigh or subsonic surface wave) is a linear combination, generally speaking, of three non-uniform plane waves, propagating in a specified direction along the boundary move slowly than all uniform plane waves. It is difficult to investigate the dispersion equation for anisotropy of general form, characterized by 21 elastic parameters. The existence and uniqueness (and also the non-existence and non-uniqueness) of Rayleigh waves was reported on the basis of a numerical analysis for different directions and different types of anisotropy (see, for example, Ref. 3). The first general result was obtained by Barnett and Lothe,[4] who established, in the case of arbitrary anisotropy, the existence of a unique Rayleigh wave (apart from a constant factor) for any direction of propagation along the surface. An additional "generic condition"[4] arose for the first time: the slowest uniform plane wave in a given direction should not satisfy the stress-free condition.

The proof, outlined in Ref. 4 and then in more detail in Ref. 5–7, is based on the "Stroh formalism",[8] i.e., on specific examples of the investigation of a naturally occurring algebraic system. The Stroh formalism was the main instrument of the theory of surface waves in isotropic elastic, piezoelectric magnetostrictive, etc. media (see, for example, the reviews in Refs 7 and 9). Its relation to the basic methods of modern mathematical physics, which rest on variational ideas, is not being considered. Recently there has been increased interest in finding an alternative to the Stroh formalism (see, for example, Ref. 10, where a new proof of the uniqueness of the Rayleigh wave is given).

In this paper we apply classical considerations, related to the principle of a minimum of the energy functional,[11] which goes back to Rayleigh, to the problem of surface waves. We describe the principles of the variational approach to analysing Rayleigh waves and, as an example of its application, we give a new proof of the Barnett - Lothe theorem of the existence of a Rayleigh wave for the case of arbitrary anisotropy. Unlike the Stroh formalism, in which the horizontal component of the slowness vector plays the role of the spectral parameter, and a basically difficult eigenvalve problem arises for the quadratic operator beam, an eigenvalue problem of the classical form is investigated in which the frequency is the spectral parameter. Hence, the approach used possesses much greater generality than the Stroh formalism. It can be applied, in principle, to inhomogeneous media (in particular media that are periodic with respect to the longitudinal variable and with respect to the depth), and also enables the existence of a Rayleigh-type wave travelling along the edge of an elastic wedge[13] to be proved.

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The central object in this approach is an ordinary differential operator in the variable characterizing the depth. The existence of a Rayleigh wave, equivalent to the presence in this operator of an eigenvalue in a certain interval, is proved by variational methods. The presence in the operator of a continuous spectrum, related to uniform plain waves, leads to difficulties of a technical nature (similar to those encountered earlier in Refs 14, 15 and 16). Here it is important that the condition that the boundary should be stress-free belongs to the class of conditions called "natural conditions" in the variational calculus. Further, after formulating the problem and the necessary factors from the theory of plane waves, we consider the homogeneous problem and we costruct, as a Friedrichs expansion, the corresponding differential operator, self-conjugate to the operator $\mathbb{L}$. We establish that its continuous spectrum lies to the right of a certain positive number $\rho\omega_*^2$, which is defined in terms of uniform plane waves. Finally, we prove that the operator $\mathbb{L}$ has at least one eigenvalue inside the interval $(0, \rho\omega_*^2)$. The presence of an eigenvalue can easily be reformulated as the existence of a Rayleigh wave.

## 2. The equation and boundary conditions

Using the linear theory, we will consider elastic waves in a homogeneous anisotropic half-space with a free boundary. The points are characterized by Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$, $x_3 \equiv z$). The displacements $\mathbf{U}(\mathbf{x}) = (U_1(\mathbf{x}), U_2(\mathbf{x}), U_3(\mathbf{x}))$, harmonic with an angular frequency $\omega > 0$, are described by the equations of the theory of elasticity

$$\mathbf{LU} + \rho\omega^2\mathbf{U} = 0 \tag{2.1}$$

in a half-space $z > 0$; $\rho = \text{const} > 0$ is the volume density of the medium. The elements of the matrix of the differential operator $\mathbf{L}$ are defined by the differential expressions

$$L_{ps} = c_{pqrs}\partial_q\partial_r, \quad p, s = 1, 2, 3, \quad \partial_m \equiv \partial/\partial x_m \tag{2.2}$$

Here $c_{pqrs}$ are real elastic constants, which possess the usual symmetry[3]

$$c_{pqrs} = c_{pqrs} = c_{rspq} \tag{2.3}$$

and which satisfy the condition that the strain energy is positive,[3] i.e.,

$$c_{pqrs}f_{pq}\overline{f_{rs}} \geq Cf_{pq}\overline{f_{pq}} \tag{2.4}$$

for a certain $C > 0$, for any non-zero symmetric tensor with components $f_{pq}$, where the bar denotes complex conjugation, and summation from 1 to 3 is carried out over repeated subscript Latin indices. The condition for the boundary to be stress-free is satisfied, namely,

$$\mathbf{TU}|_{z=0} = 0, \quad \mathbf{T} = \|T_{qr}\|, \quad T_{qr} = c_{3qrs}\partial_s, \quad q, r = 1, 2, 3 \tag{2.5}$$

## 3. Uniform and non-uniform plane waves

Plane waves, i.e., the solutions of Eq. (2.1)

$$\mathbf{U} = \mathbf{a}\exp(ik_1x_1 + ik_2x_2 + ik_3x_3), \tag{3.1}$$

in the whole of space, will play an important role in what follows. The wave vector $\mathbf{k} = (k_1, k_2, k_3)$ and the polarization vector $\mathbf{a} = (a_1, a_2, a_3)$ are constant and, generally speaking, complex. The solutions with real $\mathbf{k}$ are called uniform plane waves, and solutions with complex $\mathbf{k}$ are called non-uniform plane waves.

Substitution of expressions (3.1) into Eq. (2.1) gives the following system of linear equations

$$\mathbf{Aa} = \Lambda\mathbf{a}; \quad \Lambda = \rho\omega^2 \tag{3.2}$$

where $\mathbf{A} = \mathbf{A}(\mathbf{k})$ is a matrix with elements

$$A_{mn} = c_{mpsn}k_pk_s \tag{3.3}$$

For a real wave vector $\mathbf{k}$ the matrix a is real and, as a consequence of condition (2.4), is positive definite. The condition for system (3.2) to be solvable, i.e., the dispersion relation

$$\det\|A_{mn}(\mathbf{k}) - \delta_{mn}\rho\omega^2\| = 0$$

is a sixth-order algebraic equation in $\omega$ for specified $\mathbf{k}$; it is difficult to investigate it in the general case.

For simplicity, we will henceforth consider the propagation of waves along the $x_1$ axis only, by putting

$$k_2 = 0 \tag{3.4}$$

while $k_1$ will take only real positive values.

We will fix $\omega$. The vector $\mathbf{s} = \mathbf{k}/\omega$, which is obviously independent of $\omega$, is called the slowness vector of the plane wave.[3] A consequence of condition (3.4) is $\mathbf{s} = (s_1, 0, s_3)$. The quantity $s_1 = k_1/\omega$ is called the horizontal slowness. Uniform plane waves do not exist for fairly large values of the horizontal slowness $k_1/\omega$ (this follows, for example, from the bounded nature of the slowness surface, see Ref. 3). If $s_1^* = k_1^*/\omega$ is the largest value of the horizontal slowness for uniform plane waves, then, obviously, plane waves with $k_1 > k_1^*$ are non-uniform, i.e., they attenuate or grow as z increases. Growing solutions are of no interest here.

## 4. The traditional approach

We fix ω. A Rayleigh wave, attenuating exponentially with depth z, can be sought in the form of the sum of three non-uniform plane waves

$$\mathbf{u} = \sum_{j=1}^{j=3} \mu^{(j)} \mathbf{a}^{(j)} \exp(ik_m^{(j)} x_m) = \exp(ik_1 x_1) \sum_{j=1}^{j=3} \mu^{(j)} \mathbf{a}^{(j)} \exp(ik_3^{(j)} z) \tag{4.1}$$

with wave vectors $\mathbf{k}^{(1)}$, $\mathbf{k}^{(2)}$ and $\mathbf{k}^{(3)}$, for which the horizontal slownesses are the same and equal to $k_1/\omega$, $k_1 > k_1^*$; $\boldsymbol{k}^{(j)} = (k_1, 0, k_3^{(j)})$. Here $\mathbf{a}^{(j)}$ are the normalized polarization vectors and $\mu^{(j)}$ are the required coefficients. When $\operatorname{Im} k_3^{(j)} > 0\,(j = 1, 2, 3)$ expression (4.1) increases exponentially as z increases. The existence of a Rayleigh wave is equivalent to the possibility of obtaining a value of $k_1$ in the range $(0,\ k_1^*)$, such that the quantity (4.1) satisfies the stress-free condition (2.5). Then $c_R = \omega/k_1$ will be the phase velocity of the Rayleigh wave.

In this approach the quantity $k_1$, which occurs in the problem in question in a quadratic form, plays the role of the spectral parameter. No effective general methods of investigating these problems are known. The algebraic system for $k_1$ was investigated in a virtuoso manner using an artificial method in Refs 4 and 6.

We will use a different approach here: we will fix $k_1$, choose ω as the spectral parameter and arrive at an eigenvalue problem of classical form, which we will investigate using standard approaches.

## 5. The minimum frequency of uniform waves along the $x_1$ axis

We will consider system (3.2) as a problem of determining the eigenvalues $\Lambda = \rho\omega^2$, in which $k_1$ and $k_3$ are parameters.

We fix the real $k_1$ and we will confine ourselves for the present to real $k_3$. We will denote the eigenvalues of the matrix $\mathbf{A}(\mathbf{k})$ (they are obviously positive) by $\Lambda^{(1)}(k_3)$, $\Lambda^{(2)}(k_3)$, $\Lambda^{(3)}(k_3)$ numbering them so that $\Lambda^{(1)}(k_3) \le \Lambda^{(2)}(k_3) \le \Lambda^{(3)}(k_3)$. The following obviously exists

$$\Lambda^* = \min_{\operatorname{Im} k_3 = 0} \{\Lambda^{(1)}(k_3)\}, \quad \Lambda^* > 0 \tag{5.1}$$

Suppose $k_3{}^*$ is the value of $k_3$ (possibly not unique), for which this minimum is reached, and $\mathbf{a}^*$ is the corresponding eigenvector of the matrix $\mathbf{A}$, which can be chosen to be real. We put

$$\omega_* = \sqrt{\Lambda^*/\rho} \tag{5.2}$$

When $\omega > \omega_*$ (for fixed $k_1$) uniform plane waves exist, while when $\omega < \omega$ they do not exist.

The expression under the summation sign in (4.1) is the solution of a certain ordinary differential equation in the variable $z$. This equation, together with the boundary condition arising from equality (2.5) produces a self-conjugate operator. It has a continuous spectrum in the range $(\rho\omega_*^2,\ \infty)$, related to uniform plane waves, and an eigenvalue inside the interval $(0,\ \rho\omega_*^2)$. The corresponding eigenfunction decreases exponentially as $z \to \infty$ and describes a Rayleigh wave.

## 6. The one-dimensional problem and the corresponding energy quadratic form

We will consider the solution of problem (2.1), (2.5) of the form

$$\mathbf{U}(\mathbf{x}) = \exp(ik_1 x_1)\mathbf{u}(z), \tag{6.1}$$

where $k_1 > 0$ is a fixed real number. From relations (2.1) and (2.5) we obtain a boundary-value problem for the ordinary differential equation

$$-\mathscr{L}\mathbf{u}(z) + \rho\omega^2 \mathbf{u}(z) = 0 \quad \text{when} \quad z > 0 \tag{6.2}$$

$$\mathscr{T}\mathbf{u}(z) = 0 \quad \text{when} \quad z = 0 \tag{6.3}$$

Here

$$-\mathscr{L}_{pq} = c_{p33q}\frac{d^2}{dz^2} + ik_1(c_{p13q} + c_{p31q})\frac{d}{dz} - k_1^2 c_{p11q}, \quad \mathscr{T}_{pq} = c_{p33q}\frac{d}{dz} + ik_1 c_{p13q} \tag{6.4}$$

This is a typical eigenvalue problem.

Differential relations (6.2) and (6.3) generate, in a natural way, a positive self-conjugate operator in the space $L_2(0, \infty)$ of vector functions with scalar product

$$(\mathbf{f}, \mathbf{g})_{L_2(0,\infty)} = \int_0^\infty \mathbf{f}(z) \cdot \mathbf{g}(z) dz = \int_0^\infty f_j(z)\overline{g_j(z)} dz, \quad \mathbf{f} \cdot \mathbf{g} \equiv f_j \bar{g}_j \tag{6.5}$$

We will introduce into the finite infinitely differentiable vector functions, which satisfy condition (6.3), the symmetrical operator $\hat{L}$, acting in accordance with the rule $\hat{L}\mathbf{u} = \mathcal{L}\mathbf{u}$, and we will Friedrichs extend it in a standard way (see, for example, Ref. 11) up to the self-conjugate operator, which we will denote by $\mathbb{L}$.

We will relate the energy quadratic form $\mathcal{E}$ to problem (6.2), (6.3). We multiply (6.2) by $\bar{\mathbf{u}}$ and, integrating by parts, we obtain

$$(\mathcal{L}\mathbf{u}, \mathbf{u}) = \mathcal{E}(\mathbf{u}, \mathbf{u}) - (\mathcal{T}\mathbf{u}, \mathbf{u})|_{z=0} \tag{6.6}$$

where

$$\mathcal{E}(\mathbf{u}, \mathbf{u}) = \int_0^\infty e(k_1, \mathbf{u}, \mathbf{u}) dz \tag{6.7}$$

and the quantity

$$e(k_1, \mathbf{u}, \mathbf{u}) = c_{j33m}\partial_3 u_j \overline{\partial_3 u_m} + k_1^2 c_{j11m} u_j \overline{u_m}, \quad \partial_3 = \partial/\partial z \tag{6.8}$$

is obtained from the doubled potential energy density $c_{jklm}\partial_j u_k \overline{\partial_l u_m}$ by replacing $\partial_1$ by $ik_1$ (terms, linear in $k_1$, cancel out by virtue of symmetry conditions (2.3)).

The form $\mathcal{E}$ (6.7), (6.8) can be extended to the space $H^1(0, \infty)$ of vector functions, quadratically symmetrical together with all the first derivatives. This follows from the one-dimensional analogue of the Korn inequality

$$\mathcal{E}(\mathbf{u}, \mathbf{u}) + \|\mathbf{u}\|_{L_2(0, \infty)} \geq C\|\mathbf{u}\|_{H^1(0, \infty)}, \quad \mathbf{u}, \upsilon \in L_2(0, \infty), \quad \mathcal{E}(\mathbf{u}, \mathbf{u}) < +\infty \tag{6.9}$$

(which can easily be derived from the classical Korn inequality, see Ref. 17, or verified directly in this simple case). The self-conjugate operator, corresponding to the closed positive form $\mathcal{E}$, which we will denote by $\mathbb{L}$, is also a Friedrichs extension[11] of the operator $\hat{L}$.

## 7. The variational principle

The lower limit $\sigma(\mathbb{L})$ of the spectrum (in the case considered, this is the least eigenvalue) of the positive operator $\mathbb{L}$ can be found from the variational principle[18], which goes back to Rayleigh,

$$\underline{\sigma(\mathbb{L})} = \inf\left\{\frac{(\mathbb{L}\mathbf{u}, \mathbf{u})}{(\mathbf{u}, \mathbf{u})}\right\}, \quad \mathcal{T}(\mathbf{u})|_{z=0}; \quad \mathbf{u} \in H^1(0, \infty) \tag{7.1}$$

($\mathbf{u} \neq 0$ belongs to the region of definition of $\mathcal{E}$). For boundary conditions (6.3), which belong to the class of conditions called natural conditions in the variational calculus, we can use the classical form of the Rayleigh variational principle

$$\underline{\sigma(\mathbb{L})} = \inf\left\{\frac{(\mathcal{E}\mathbf{u}, \mathbf{u})}{(\mathbf{u}, \mathbf{u})}\right\}, \quad \mathbf{u} \in H^1(0, \infty) \tag{7.2}$$

Unlike functional (7.1), here it is not required to satisfy boundary conditions (6.3).

We will explain this point, which will be important later. We will assume that the minimum of functional (7.1) is reached on the twice differentiable function $\mathbf{u}_0$ and is equal to $\rho\omega^2$. This is equivalent to the condition that the first variation of functional (7.1) vanishes, integrating by parts in which we have

$$\mathcal{E}(\mathbf{u}_0, \upsilon) - \rho\omega^2(\mathbf{u}_0, \upsilon) = \mathcal{T}\mathbf{u}_0 \cdot \bar{\upsilon}|_{z=0} + \int_0^{+\infty} (\mathcal{L}\mathbf{u}_0 - \rho\omega^2\mathbf{u}_0) \cdot \bar{\upsilon} dz = 0 \tag{7.3}$$

Here $\mathcal{E}(\mathbf{u}, \boldsymbol{v})$ is a bilinear form, obtained from the quadratic form of the energy (6.7) by replacing $\bar{\mathbf{u}}$ by $\bar{\boldsymbol{v}}$, while the function $\boldsymbol{v} = \boldsymbol{v}(z)$ is arbitrary ($\boldsymbol{v} = H^1(0, \infty)$), which, as is usual for the variational calculus,[11] implies satisfaction not only of the equation $\mathcal{L}\mathbf{u}_0 - \rho\omega^2\mathbf{u}_0 = 0$ but also of the boundary condition (6.3). In this discussion, which is typical for the variational calculus, the need for the function $\mathbf{u}$ to be smooth, in order to carry out integration by parts, is not substantiated (otherwise, this proof would be quite simple, see Ref. 11) and the existence of the element $\mathbf{u}_0$ on which an infimum is realized, is not proved. The proof of this, which is complicated by the presence of a continuous spectrum in the operator $\mathbb{L}$, is given below.

## 8. The continuous spectrum of the operator $\mathbb{L}$

It is natural to expect that the quantity $\rho\omega_*^2$, introduced by equalities (5.1) and (5.2) when considering uniform plane waves, will be the lower limit of the continuous spectrum of the one-dimensional operator $\mathbb{L}$.

**Theorem 1.** *The interval* $(-\infty, \rho\omega_*^2)$ *does not contain points of the continuous spectrum of the operator* $\mathbb{L}$.

The proof of this assertion is based on the following estimate.

**Lemma 1.** *The following inequality holds for any vector-function* $\mathbf{u} = \mathbf{u}(z) \in H^1(0, \infty)$

$$\mathcal{E}(\mathbf{u}, \mathbf{u}) \geq \rho\omega_*^2 \int_0^\infty |\mathbf{u}|^2 dz - C_1 \int_0^2 |\mathbf{u}|^2 dz \tag{8.1}$$

where the constant $C_1$ is independent of $\mathbf{u}$.

Proof. Suppose $\chi_1 = \chi_1(z)$ and $\chi_2 = \chi_2(z)$ are smooth real truncating functions, defined for $-\infty < z < +\infty$, such that $\chi_1(z) = 0$ when $z < 1$, $\chi_2(z) = 0$ when $z > 2$, and

$$\chi_1^2(z) + \chi_2^2(z) \equiv 1 \tag{8.2}$$

Henceforth denoting a derivative with respect to z by a prime, we note that

$$u_j'\overline{u_m'} = (\chi_1^2 + \chi_1^2)u_j'\overline{u_m'} = \sum_{\gamma=1}^{\gamma=2}[(\chi_\gamma u_j)'(\chi_\gamma \overline{u_m})' - \chi_\gamma\chi_\gamma'(u_j'\overline{u_m} + u_j\overline{u_m'}) - (\chi_\gamma)^2 u_j\overline{u_m}] \tag{8.3}$$

As a consequence of identity (8.2)

$$\chi_1\chi_1' + \chi_2\chi_2' = [(\chi_1)^2 + (\chi_2)^2]'/2 = 0$$

and the middle term on the right-hand side of (8.3) vanishes in the summation. Using this, we can easily verify the identity

$$\mathcal{E}(\mathbf{u}, \mathbf{u}) = \sum_{\gamma=1}^{\gamma=2}\left(\mathcal{E}(\chi_\gamma\mathbf{u}, \chi_\gamma\mathbf{u}) - \int_0^\infty c_{j33m}(\chi_\gamma')^2 u_j\overline{u_m}dz\right) = \sum_{\gamma=1}^{\gamma=2}\mathcal{E}(\chi_\gamma\mathbf{u}, \chi_\gamma\mathbf{u}) - \int_0^\infty \chi\mathbf{C}\mathbf{u}\cdot\mathbf{u}dz$$

where $\chi = \chi(z) = \chi_1'^2 + \chi_2'^2$ is a positive function and $\mathbf{C}$ is a symmetric positive matrix (its positiveness follows from condition (2.4)) with elements $C_{pq} = C_{p33q}$. Hence we have the estimate

$$\mathcal{E}(\mathbf{u}, \mathbf{u}) \geq \mathcal{E}(\chi_1\mathbf{u}, \chi_1\mathbf{u}) - C_2\int_0^\infty |\mathbf{u}|^2 dz \tag{8.4}$$

with a constant $C_2$ that is independent of $\mathbf{u}$.

We will estimate the first term on the right-hand side of inequality (8.4). We will put $\mathbf{w}(z) = \chi_1(z)\mathbf{u}(z)$, supplement $\mathbf{w}(z) \equiv 0$ for $z < 0$, and introduce the following Fourier transformation $z \to k_3$

$$\hat{\mathbf{w}}(k_3) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathbf{w}(z)\exp(ik_3 z)dz$$

Obviously, $\hat{\mathbf{w}}'(k_3) = ik_3\hat{\mathbf{w}}(k_3)$. Using expression (6.8) and Parseval's equality

$$\int_{-\infty}^{+\infty} w_p\overline{w}_s dz = \int_{-\infty}^{+\infty}\hat{w}_p\overline{\hat{w}}_s dk_3$$

we obtain

$$\mathcal{E}(\chi_1\mathbf{u}, \chi_1\mathbf{u}) = \int_{-\infty}^{+\infty} e(k_1, \mathbf{w}, \mathbf{w})dz = \int_{-\infty}^{+\infty} c_{pqrs}k_q k_r\hat{w}_p\overline{\hat{w}}_s dk_3 = \int_{-\infty}^{+\infty}\mathbf{A}\hat{\mathbf{w}}\cdot\overline{\hat{\mathbf{w}}}dk_3$$

Now we have $\mathbf{A}\hat{\mathbf{w}}\cdot\bar{\hat{\mathbf{w}}} \geq \rho\omega_*^2|\hat{\mathbf{w}}|^2$, where $\Lambda^* = \rho\omega_*^2$ in accordance with Eqs. (5.1) and (5.2). Using Parseval's equality, we obtain

$$\mathcal{E}(\chi_1\mathbf{u}, \chi_1\mathbf{u}) \geq \rho\omega_*^2\int_{-\infty}^{+\infty}\chi_1^2|\mathbf{u}|^2 dz = \rho\omega_*^2\int_0^\infty\chi_1^2|\mathbf{u}|^2 dz \geq \rho\omega_*^2\int_0^\infty|\mathbf{u}|^2 dz$$

which, together with inequality (8.4), proves Lemma 1.

We will now prove Theorem 1. Following the well-known scheme in Refs 12 and 13, we will assume the opposite. Suppose a point $\rho\omega_1^2$ of the continuous spectrum of the operator $\mathbb{L}$ exists, such that

$$\rho\omega_1^2 < \rho\omega_*^2 \tag{8.5}$$

Then[18] in the region in which $\mathbb{L}$ is defined there is an orthonormalized sequence such that $\mathbb{L}\mathbf{u}_n - \rho\omega_1^2\mathbf{u}_n \to 0$ in $L_2(0, \infty)$ and, consequently,

$$\mathcal{E}(\mathbf{u}_n, \mathbf{u}) \to \rho\omega_1^2, \quad n \to \infty \tag{8.6}$$

The sequence $\mathbf{u}_n$ is orthonormalized in $L_2(0,\infty)$, and by virtue of relations (8.6) and (6.9) terms of the sequence $\mathbf{u}_n$ are bounded in the norm $H^1(0,\infty)$. Consequently, we can choose a subsequence $\mathbf{u}_{n_p}$, which converges to zero in $L_2(0,2)$. For this we can rewrite relation (8.6) in the form

$$\mathcal{E}(\mathbf{u}_{n_p}, \mathbf{u}_{n_p}) + C\int_0^2 |\mathbf{u}_{n_p}|^2 dz \to \rho\omega_1^2, \quad n_p \to +\infty. \tag{8.7}$$

At the same time, it follows from inequality (8.1) that the left-hand side of relation (8.7) is not less than $\rho\omega_*^2$, whence $\omega_1^2 \geq \omega_*^2$, which contradicts condition (8.5).

## 9. The discrete spectrum of the operator $\mathbb{L}$

Suppose $\mathbf{a}^*$ is the eigenvector of the matrix $\mathbf{A}$, corresponding to the eigenvalue $\rho\omega_*^2$ (see Section 5). We will introduce the vector functions

$$\mathbf{U}^*(\mathbf{x}) = \mathbf{a}^*\exp(ik_1^*x_1 + ik_3^*z), \quad \mathbf{u}^*(z) = \mathbf{a}^*\exp(ik_3^*z)$$

They satisfy Eqs. (2.1) and (6.2) respectively (but in the generic case they do not satisfy the boundary conditions when $z = 0$). The equalities

$$\mathbf{T}\mathbf{U}^*|_{z=0} = 0 \quad \text{и} \quad \mathcal{T}\mathbf{u}^*|_{z=0} = 0 \tag{9.1}$$

are equivalent. It is obvious that $\mathbf{U}^*$ is a uniform plane wave, propagating in the direction $(k_1, 0, k_3)$ with the least possible frequency for a fixed value of $k_3$.

**Theorem 2.** *If*

$$\mathbf{T}\mathbf{U}^*|_{z=0} \neq 0 \tag{9.2}$$

*then the interval* $(0, \rho\omega_*^2)$ *contains* (*at least one*) *eigenvalue of the operator* $\mathbb{L}$.

By virtue of Theorem 1 it is sufficient to establish that $\underline{\sigma}(\mathbb{L}) < \rho\omega_*^2$, i.e., that the form

$$\mathcal{B}(\mathbf{u}, \upsilon) = \mathcal{E}(\mathbf{u}, \upsilon) - \rho\omega_*^2(\mathbf{u}, \upsilon) \tag{9.3}$$

is not positive. To do this we consider its values in the sequence

$$\mathbf{u}_n^*(z) = \exp(-z/n)\mathbf{u}^*(z), \quad n = 1, 2, \ldots \tag{9.4}$$

**Lemma 2.** *The following inequality holds*

$$|\mathcal{B}(\mathbf{u}_n^*, \upsilon_n^*)| \leq \text{const}|\mathbf{a}^*|^2/n, \quad \text{const} > 0 \tag{9.5}$$

For the proof we note that definitions (9.4), (9.3) and (6.8) give

$$\mathcal{B}(\mathbf{u}_n^*, \upsilon_n^*) = \int_0^\infty [e(k_1^*, \mathbf{u}_n^*, \mathbf{u}_n^*) - \rho\omega_*^2|\mathbf{u}_n^*|^2]dz \tag{9.6}$$

It follows from definition (6.8) that

$$e(k_1^*, \mathbf{u}_n^*, \mathbf{u}_n^*) = \exp(-2z/n)\{[c_{j33m}k_1^{*2} + c_{j33m}k_3^{*2}]a_j^*\overline{a_m^*} + n^{-2}c_{j33m}a_j^*\overline{a_m^*}\} \tag{9.7}$$

The sum of the terms in the braces, not containing the factor $n^{-2}$, is equal to $\mathbf{A}(\mathbf{k}^*)\mathbf{a}^* \cdot \mathbf{a}^* = \rho\omega_*^2|\mathbf{a}_n^*|^2$ and is independent of $z$. As a result, equality (9.6) takes the form

$$\mathcal{B}(\mathbf{u}_n^*, \mathbf{u}_n^*) = \int_0^\infty \exp\left(-2\frac{z}{n}\right)\frac{c_{j33m}a_j^*\overline{a_j^*}}{n^2}dz = \frac{\mathbf{Ca}^2 \cdot \mathbf{a}^*}{2n} \tag{9.8}$$

where $\mathbf{C}$ is a symmetric positive matrix, indicated when proving Lemma 1. The constants in inequalities (9.5) and (2.4) are obviously connected by the relation const = $C/2$.

We will now prove Theorem 2. If the form $\mathcal{B}(\cdot, \cdot)$ is positive, then, for any vector functions $\mathbf{u} = \mathbf{u}(z)$ and $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}(z)$ from $H^1(0, \infty)$, the Cauchy - Bunyakovskii inequality holds, i.e.,

$$|\mathcal{B}(\mathbf{u}, \upsilon)|^2 \leq 4\mathcal{B}(\mathbf{u}, \mathbf{u})\mathcal{B}(\upsilon, \upsilon)$$

While substituting $\mathbf{u} = \mathbf{u}_n^*$ here and using inequality (9.5) it can be shown that

$$\mathcal{B}(\mathbf{u}_n^*, \upsilon) \leq \text{const}/n \tag{9.9}$$

At the same time, integration by parts in Eq. (9.3) gives

$$\mathcal{B}(\mathbf{u}_n^*, \upsilon) = \mathcal{T}\mathbf{u}_n^* \cdot \bar{\upsilon}|_{z=0} + \int_0^{+\infty} (\mathcal{L}\mathbf{u}_n^* - \rho\omega_*^2 \mathbf{u}_n^*) \cdot \upsilon dz \tag{9.10}$$

It follows from inequality (9.9), by virtue of the arbitrary nature of $\boldsymbol{\upsilon}$ in Eq. (9.10), that

$$\mathcal{T}\mathbf{u}_n^*|_{z=0} \to 0, \quad n \to +\infty$$

and from relations (9.4) and (6.4) we have

$$\mathcal{T}\mathbf{u}_n^*|_{z=0} = \mathcal{T}\mathbf{u}^*|_{z=0} + \frac{\mathbf{Cu}^*}{n} \to \mathcal{T}\mathbf{u}^*|_{z=0}, \quad n \to +\infty$$

Hence, $\mathcal{T}\mathbf{u}^*|_{z=0} = 0$, which, due to the equivalence of Eqs. (9.1), contradicts the condition of the theorem. Hence, the form $\mathcal{B}(\cdot, \cdot)$ is not positive. Consequently, the functional $\mathcal{E}(\mathbf{u}, \mathbf{u})/(\mathbf{u}, \mathbf{u})$ takes values in the range $(0, \rho\omega_*^2)$. It follows from this[18] that the operator $\mathbb{L}$ has a spectrum (discrete, in view of Theorem 1) in the range $(0, \rho\omega_*^2)$.

The theorem is proved.

We have thus obtained the following result.

Suppose $s_1^* = k_1^*/\omega > 0$ is the greatest horizontal slowness of uniform plane waves with wave number $\mathbf{k} = (k_1, 0, k_3)$. If the generic condition is satisfied, i.e., the plane wave with the last phase velocity $\mathbf{U}^*$ (9.1) does not satisfy the condition for the boundary to be stress-free (2.5), a Rayleigh wave of the form (4.1) exists with horizontal slowness $1/c_R = s_1$, which satisfies the condition $s_1 > s_1^*$.

The condition for the phase fronts to propagate along the $x_1$ axis (3.4) plays no role, and the wave vector of the form $\mathbf{k} = (k_1, 0, k_3)$ can be replaced by any vector $\mathbf{k} = (k_1, k_2, k_3)$.

The Barnett - Lothe existence theorem has been proved using only fundamental ideas of modern mathematical physics and the apparatus of the theory of operators.


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